Journal of Economic Dynamics and Control
23 (1998) 277–301

JOURNAL OF
Economic
Dynamics
& Control

Optimal spreading when spreading is optimal

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Received 13 May 1997; accepted 2 December 1997

Abstract

This paper assumes an investor who has a non-traded position operating in a stochastic interest rates environment. The investor trades continuously either distinct futures contracts or distinct forward contracts in order to maximize his expected utility of terminal wealth. In order to reach the welfare level of the first best optimum, the investor must incorporate into his portfolio either two distinct futures contracts or two distinct forward contracts. The optimal forward contracts dynamic spreading strategy has two components, a speculative component and a minimum-variance hedging component. The minimum-variance hedging component is composed of a short position in the nearby contract and a long position in the deferred contract. The speculative component serves to replicate the growth optimum portfolio. The speculative component is composed of a short position in the contract which is the most negatively correlated with the growth optimum portfolio and a long position in the other contract. The marking-to-market procedure of the futures positions forces the investor to hold less futures contracts than the corresponding forward contract positions. The analysis is also extended to incomplete markets and to inter-market spreading. © 1998 Elsevier Science B.V. All rights reserved.

JEL classification: G11; G13; G15

Keywords: Spreading; Forward contracts; Marking-to-market; Martingale-approach

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PII S0165-1889(97)00120-6

1. Introduction

The bulk of intertemporal futures hedging literature[1] has dealt with an investor who is endowed with a non-traded cash position and is allowed to trade futures contracts continuously. When the dynamics of the futures settlement price and the dynamics of the underlying asset price are perfectly correlated, the investor can achieve the welfare level of the first best optimum (Adler and Detemple, 1988a). This welfare level is equal to the one the investor would have reached had he been allowed to trade the primitive assets freely. The optimal futures trading strategy has the traditional decomposition into a speculative component and a minimum-variance hedging component (Anderson and Danthine, 1983). When the opportunity set is stochastic, the futures strategy includes (except for a Bernoulli investor) a Merton/Breeden hedging component which allows the investor to hedge against the fluctuations of the opportunity set.

When the correlation between the futures settlement price dynamics and the underlying asset price dynamics is imperfect, the investor cannot reach the first best optimum welfare level. He can only arrive at the welfare level of the second best optimum (Adler and Detemple, 1988a). In particular, if the investor is a pure hedger, he cannot reach a perfect hedge of his non-traded position. Therefore, in this case, the opportunity set generated by the futures markets is different than the one generated by the primitive assets markets.

The imperfect correlation case is important since it is frequently faced by investors. When the futures contract is written on an asset other than the asset of the non-traded position, there will generally be an imperfect correlation between the dynamics of the futures settlement price and the dynamics of the asset of the non-traded position. Hence, using futures contracts in order to cross-hedge the risk resulted from the non-traded position does not allow the investor to reach the welfare level of the first best optimum. Even if the futures contract is written on the asset of the non-traded position, the dynamics of the futures settlement price and the dynamics of the underlying asset price will be imperfectly correlated in the presence of stochastic interest rates.

One cannot conclude that in the case of imperfect correlation, the constrained investor will always be worse off compared to the case of perfect correlation. As is shown in this paper, even in the case of imperfect correlation, an investor can reach the welfare level of the first best optimum. This can be attributed to the richness of the futures markets where futures contracts on the same underlying asset are traded for several maturities. Investors who hold simultaneous position

[1] See, for example, Adler and Detemple (1988a), Adler and Detemple (1988b), Duffie and Jackson (1990), Duffie and Richardson (1991), Schweizer (1992), Lioui and Poncet (1996). Another branch address the issue of the firm's optimal hedging policies and their determinants (see Ho, 1984; Stulz, 1984; Smith and Stulz, 1985; Eldor and Zilcha, 1987; Zilcha and Eldor, 1991).

in different futures contracts (intra-market spreading) are able to span the opportunity set of the primitive assets and thus are able to reach the first best optimum welfare level.[2]

A strategy that involves a simultaneous position in distinct futures or forward contracts on the same underlying asset is called a futures spread or a forward spread. The spreading literature in futures or forward contracts has mainly dealt with arbitrageurs, i.e., specific profitable trading strategies that combine several futures contracts (Wahab, 1995 and references therein). The theoretical analysis of optimal spreading has been somewhat limited. Schrock (1971) used a traditional mean-variance static framework in order to analyze the optimal spreading strategy using futures or forward contracts. He established the fact that spreading strategies are important because they expand the feasible set of investment opportunities beyond the set defined by out-writing short and long positions. Poitras (1989) derived similar results using a quadratic utility function. Peterson (1977) added an economic rationale for trading futures spreads by postulating that the margin requirements of futures held in spreads are lower than margin requirements on futures contracts held in out-writing long or short positions. Da-Hsiang-Donald (1992) used an econometric model and derived the spreading strategy for an investor (a hedger) endowed with a non-traded position. Unfortunately, the author assumed that the term structure of interest rates is flat and equal to zero. This assumption implies[3] that the futures contract price and the underlying asset price are perfectly correlated. Moreover, the prices of futures contracts with different maturities are perfectly correlated. Hence, there is not a unique solution to the spreading strategy of an investor seeking to minimize the instantaneous risk of his hedged portfolio. The author's results are due to the econometric models postulated a priori for the futures price and for the underlying asset's price dynamics.

In this paper, the primitive asset market is complete and the derivative contracts are arbitrage free priced. Spreading is optimal in our framework since

[2] We do not attempt to solve for the optimal design of futures contracts which allows an efficient allocation of risk. The work of Duffie and Rahi (1995) gives an excellent outlook of the literature on this issue. In a general equilibrium approach, futures innovation intends to complete the primitive asset market which is assumed to be incomplete. Had the primitive assets market been complete, redundant futures contracts will not be traded. Elul (1995) has shown that by completing the financial market, some investor's could be worse off since futures trading alter the equilibrium relative prices of the assets. Hence, futures innovation should not necessarily aim for completing the market and there may exist an optimal level of market incompleteness Ohashi, 1995). In the framework of the intertemporal hedging literature, the spanning function of the futures contracts has quite a different role. The futures contracts are used in order to span an opportunity set as the one spanned by the primitive assets or a subset of this opportunity set. Hence, even if they are redundant, futures contracts are Pareto improving for constrained investors.

[3] This argument is true only when the dividend yield is deterministic, which is the case in Da-Hsiang-Donald (1992).

the constrained investor faces a complete market (as if he trades the primitive assets) which allows him to reach the welfare level of the first best optimum. This spanning property of the derivative contracts which improve the hedging effectiveness for an investor endowed with a non-traded cash position could be seen as an economic rationale for the existence of several futures contracts of different maturities on the same underlying asset.

When interest rates are deterministic, it is well known that forward prices and the corresponding futures settlement price are identical. Nevertheless, for an investor seeking to hedge a non-traded position, the forward contract strategy and the futures contract strategy are different under deterministic interest rates. The marking-to-market of the futures positions forces the hedger to tail his hedge when using futures contracts until the point in time the hedge is lifted (Figlewski et al., 1991). Under stochastic interest rates, futures contracts settlement prices and the corresponding forward contracts prices are different. Although futures markets are more common than forward markets when the underlying asset is a stock index or a commodity, by solving first in this paper the investor's problem in the forward markets case, we are able to detect the impact of stochastic interest rates on the nature of spreading strategy. In the second stage, we isolate the effects of the marking-to-market procedure on the investor's optimal strategy by comparing the futures spreading strategy to the forward spreading strategy.

The paper is organized as follows. Section 2 describes the financial markets. In Section 3, we set the investor's problem and in Section 4, we derive the investor's optimal forward spreading strategy. The effect of marking-to-market is exhibited in Section 5 where the investor's optimal futures reading strategy is derived. In Section 6 we extend our analysis to incomplete markets and intermarket spreading. Section 7 summarizes our finding and suggests some other extensions. The appendix contains the proofs of the results.

2. The framework

The (non-constrained) agents trade continuously in a frictionless financial market until time τ, where τ is the horizon of the economy. Three long lived assets are traded, a locally riskless asset (money market account), a discount bond and a stock index. The discount bond pays one dollar at its maturity, $\tau_D < \tau$. At each time t, where $0 \leq t \leq \tau_D$, the price $P(t, \tau_D)$ of the discount bond whose maturity is τ_D is given by

$$P(t, \tau_D) = \exp\left\{ - \int_t^{\tau_D} f(t, T)\, dT \right\}, \tag{1}$$

where $f(t, T)$ is the instantaneous forward interest rate for maturity T at time t. The spot rate is $r(t) = f(t, t)$ and the price of the locally riskless asset (an interest bearing bank account) is

$$B(t) = \exp\left\{\int_0^t r(s)\,\mathrm{d}s\right\}. \tag{2}$$

Assume that the instantaneous forward interest rate is the solution to the following stochastic differential equation:[4]

$$\mathrm{d}f(t,T) = \mu(t,T)\,\mathrm{d}t + v_1\,\mathrm{d}Z_1(t) + v_2\,\mathrm{d}Z_2(t) \tag{3}$$

for $0 \leq T < \tau_{\mathrm{D}}$, where $\mu(t,T)$ is the drift term that satisfies the usual conditions[5] such that Eq. (3) has a unique solution and v_1 and v_2 are strictly positive constants. $Z_1(t)$ and $Z_2(t)$ are two independent Brownian motions defined on a complete probability space (Ω, F, Q) where Ω is the state space, F is the σ-algebra representing measurable events, and Q is the historical probability measure. The instantaneous forward rate is adapted to the augmented filtration generated by the two Brownian motions. This filtration which is denoted by $\boldsymbol{F} \equiv \{F_t\}_{t\in[0,\tau]}$ satisfies the usual conditions.[6] $f(0,T)$, the initial value of the instantaneous forward rate, is given by the initial term structure of interest rates which prevails in the market.

Substituting for the instantaneous forward rate (3) in Eq. (1) and applying Ito's lemma, one obtains[7]

$$\begin{aligned}\mathrm{d}P(t,\tau_{\mathrm{D}}) = P(t,\tau_{\mathrm{D}})[[b(t,\tau_{\mathrm{D}}) + r(t)]\,\mathrm{d}t - v_1(\tau_{\mathrm{D}} - t)\,\mathrm{d}Z_1(t)\\ - v_2(\tau_{\mathrm{D}} - t)\,\mathrm{d}Z_2(t)],\end{aligned} \tag{4}$$

where

$$b(t,\tau_{\mathrm{D}}) = -\int_t^{\tau_{\mathrm{D}}} \mu(t,T)\,\mathrm{d}T + \frac{1}{2}v_1^2(\tau_{\mathrm{D}} - t)^2 + \frac{1}{2}v_2^2(\tau_{\mathrm{D}} - t)^2.$$

The stock index pays a continuous dividend yield δ. The stock index price, denoted by $I(t)$, solves the following stochastic differential equation:

$$\mathrm{d}I(t) = (\psi(t,I(t)) - \delta)I(t)\,\mathrm{d}t + \sigma_1 I(t)\,\mathrm{d}Z_1(t), \quad I(0) = x > 0, \tag{5}$$

[4] The suggested model for the instantaneous forward rate allows this economic variable to take negative values. Unfortunately, this is one of the few polar cases in Financial Economics which allows for explicit results. For a lucid discussion of this issue, see Subrahmanyam (1996).

[5] See conditions C.1 p. 80 and C.2 p. 81 of Heath et al. (1992).

[6] The σ-algebra contains the events whose probability with respect to Q is null. See Karatzas and Shreve (1991) (p. 89).

[7] To obtain expression (4) and the expression for $b(\ )$, we have followed step by step the approach in Heath et al. (1992) (p. 82, Eq. (8)).

where $\psi(t, I(t))$ is the total index yield (dividend plus capital gains) and $\sigma_1 > 0$ is the constant volatility. $\psi(t, I(t))$ satisfies the usual conditions[8] such that Eq. (5) has a unique (strong) solution.

Each (non-constrained) agent follows a portfolio strategy which consists of the locally riskless asset, the discount bond and the stock index. These strategies are assumed to be admissible strategies.[9]

Finally, we assume that there are no arbitrage opportunities on the financial markets. Following Harrison and Kreps (1979), this is equivalent[10] to the existence of a probability measure equivalent to Q such that the (with respect to a suitable numéraire) discounted price process plus the cumulated discounted dividends are martingales. The price of any attainable contingent claim is simply the conditional expectation of its future cash flows in units of the numéraire with respect to this martingale measure.

Let us construct the martingale measure on the financial market constituted of the three assets when the numéraire is the locally riskless asset $B(\ )$. Define

$$\kappa(t) \equiv \begin{pmatrix} \kappa_1(t) \\ \kappa_2(t) \end{pmatrix} = -\begin{pmatrix} \sigma_1 & 0 \\ -\nu_1(\tau_D - t) & -\nu_2(\tau_D - t) \end{pmatrix}^{-1} \begin{pmatrix} \psi(t, I(t)) - r(t) \\ b(t, \tau_D) \end{pmatrix} \quad (6)$$

and assume that

$$E^Q\left[\exp\left\{\frac{1}{2}\int_0^{\tau_D} (\kappa_1^2(t) + \kappa_2^2(t))\,dt\right\}\right] < \infty. \quad (7)$$

Now, define the new stochastic process

$$\eta(t) = \exp\left\{\int_0^t \kappa_1(s)\,dZ_1(s) + \int_0^t \kappa_2(s)\,dZ_2(s) - \frac{1}{2}\int_0^t (\kappa_1^2(s) + \kappa_2^2(s))\,ds\right\}. \quad (8)$$

This is strictly positive continuous local martingale and Novikov's condition (7) ensures that it is in fact a martingale with expectation equal to 1 with respect to Q.[11] Hence, we can define a new probability measure equivalent

[8] See Karatzas and Shreve (1991) (p. 285).

[9] To save space, we shall not specify the properties of the admissible strategies. The complete definition given by Cox and Huang (1989) is the one adopted.

[10] In fact, this equivalence result holds only for the simple strategies, i.e., strategies which need a portfolio reallocation only a finite number of times (see Harrison and Kreps, 1979; Harrison and Pliska, 1981). Property 5 of Cox and Huang (1989) definition of admissible strategies excludes arbitrage opportunities for non simple strategies as proved by Dybvig and Huang (1988). For further results, see Delbaen and Schachermayer (1994).

[11] See Karatzas and Shreve (1991) (p. 198).

to Q such that

$$\left.\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q}\right|_{F_t} = \eta(t) \tag{9}$$

for $t < \tau_{\mathrm{D}}$.[12]

Using Girsanov's theorem,[13] the following two processes:

$$\begin{aligned} \tilde{Z}_1(t) &= Z_1(t) - \int_0^t \kappa_1(s)\,\mathrm{d}s, \\ \tilde{Z}_2(t) &= Z_2(t) - \int_0^t \kappa_2(s)\,\mathrm{d}s, \end{aligned} \tag{10}$$

are two independent Brownian motions with respect to $\tilde{Q}$. The dynamics of the discount bond and of the stock index prices are as follows:

$$\mathrm{d}P(t,\tau_{\mathrm{D}}) = P(t,\tau_{\mathrm{D}})[r(t)\,\mathrm{d}t - v_1(\tau_{\mathrm{D}} - t)\,\mathrm{d}\tilde{Z}_1(t) - v_2(\tau_{\mathrm{D}} - t)\,\mathrm{d}\tilde{Z}_2(t)], \tag{11}$$

$$\mathrm{d}I(t) = (r(t) - \delta)I(t)\,\mathrm{d}t + \sigma_1 I(t)\,\mathrm{d}\tilde{Z}_1(t). \tag{12}$$

Moreover,

$$\frac{P(t,\tau_{\mathrm{D}})}{B(t)} \quad \text{and} \quad \frac{I(t)}{B(t)} + \int_0^t \frac{\delta I(s)}{B(s)}\,\mathrm{d}s$$

are martingales with respect to $\tilde{Q}$. Therefore, we arrive at the pricing function for contingent claims, namely, the expectation of their cash flows expressed in terms of the riskless asset under the measure $\tilde{Q}$.

Now to complete the exposition, let us find the dynamics of the spot rate with respect to the martingale measure $\tilde{Q}$. Following Heath et al. (1992),[14] one obtains

$$\mu(t,T) = v_1^2(T-t) + v_2^2(T-t) - v_1\kappa_1(t) - v_2\kappa_2(t) \tag{13}$$

for $T \leqslant \tau_{\mathrm{D}}$. Substituting Eq. (13) in Eq. (3), and using Eq. (10) one obtains

$$f(t,T) = f(0,T) + (v_1^2 + v_2^2)\,t\left(T - \frac{t}{2}\right) + v_1\tilde{Z}_1(t) + v_2\tilde{Z}_2(t) \tag{14}$$

[12] Karatzas (1997) (p. 4) gives an excellent exposition on the construction of a martingale measure in our setting.

[13] See Karatzas and Shreve (1991) (p. 190).

[14] Proposition 3, p. 86 of Heath et al. (1992). See also Musiela and Rutkowski (1997) (p. 297).

and, therefore,

$$r(t) = f(t,t) = f(0,t) + (v_1^2 + v_2^2)\frac{t^2}{2} + v_1\tilde{Z}_1(t) + v_2\tilde{Z}_2(t). \tag{15}$$

In the next section we present the investor's problem.

3. The investor's problem

The investor is endowed with a non-traded cash position of $\pi > 0$ units of the stock index. He continuously trades the riskless asset and two forward contracts with different maturities, written on the stock index. The investor is an expected utility maximizer.

In our framework, the price of a forward contract maturing at $\tau_i < \tau_D$ is

$$G(t,\tau_i) = \frac{I(t)\exp\{-\delta(\tau_i - t)\}}{P(t,\tau_i)}. \tag{16}$$

Denote by $\beta_i(t)$ the number of the forward contracts of maturity τ_i held by the investor at time t and $\alpha(t)$ the number of the locally riskless asset. The investor's trading strategy is chosen in such a way that its value at each time t is as follows:

$$W(t) = \pi I(t) + \int_0^t \pi\delta I(s)\,\mathrm{d}s + P(t,\tau_1)\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)$$

$$+ P(t,\tau_2)\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2) + \alpha(t)B(t). \tag{17}$$

The investor's wealth has four components (i) the investor's non-traded position, (ii) the present value of the gains/loses from the changes in prices of the forward contract of maturity τ_1 position, (iii) the present value of the gains/loses from the changes in prices of the forward contract of maturity τ_2 position and (iv) the locally riskless asset position.

The investor maximizes the expected utility of his terminal wealth at his investment horizon τ_I. In the following, we assume a Bernoulli investor, i.e., an investor endowed with a logarithmic utility function. This case is a benchmark case in Financial Economics (see Adler and Detemple, 1988a, Kuwana, 1995). Besides, this is the only case in continuous time finance where an explicit solution can be found for a general process of the market prices for risk associated with the two sources of uncertainty.

The investor's utility function is therefore:

$$u(W(\tau_I,\omega)) = \mathrm{Ln}(W(\tau_I,\omega)), \quad \omega \in \Omega \tag{18}$$

and the investor's objective is to

$$\begin{array}{ll} \max\limits_{\alpha,\beta_1,\beta_2} & E^Q[\mathrm{Ln}(W(\tau_I))] \\ \text{s.t.} & \text{The portfolio strategy is an admissible strategy} \\ & \text{which satisfies Eq. (17).} \end{array} \tag{19}$$

In the next section we derive the investor's optimal forward spreading strategy explicitly. Although futures markets are more common than forward markets when the underlying asset is a stock index or a commodity, by solving first the investor's problem in the forward markets case, we are able to detect the impact of stochastic interest rates on the nature of spreading strategy. In the second stage (in Section 5), we isolate the effects of marking-to-market on the investor's optimal strategy by comparing the futures spreading strategy to the forward spreading strategy.

4. The optimal forward spreading strategy

In this section we present the optimal dynamic forward spreading strategy that maximizes the investor's expected utility of terminal wealth. Since the investor has a Logarithmic utility function, it is well known that the investor will choose the growth optimum portfolio as an optimal solution to his optimization problem when his initial wealth is the initial value of his non-traded position. The value of his wealth at each time t will be $\pi I(0)\exp\{-\delta\tau_I\}B(t)\eta(t)^{-1}$. Hence, the forward contracts spreading strategy allows the investor to replicate this wealth taking into account the presence of a non-traded cash position. This forward contracts dynamic strategy is given in the following proposition:

Proposition 1. *The optimal trading strategy in forward contract with maturity τ_1 and forward contract with maturity τ_2 is, respectively,*

$$\begin{pmatrix} \hat{\beta}_1(t) \\ \hat{\beta}_2(t) \end{pmatrix} = \begin{pmatrix} \hat{\beta}_1^{\mathrm{s}}(t) \\ \hat{\beta}_2^{\mathrm{s}}(t) \end{pmatrix} + \begin{pmatrix} \hat{\beta}_1^{\mathrm{h}}(t) \\ \hat{\beta}_2^{\mathrm{h}}(t) \end{pmatrix}, \tag{20}$$

where

$$\begin{pmatrix} \hat{\beta}_1^{\mathrm{s}}(t) \\ \hat{\beta}_2^{\mathrm{s}}(t) \end{pmatrix} = \begin{pmatrix} \dfrac{v_2(\tau_2 - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_2 - t)]\kappa_2(t)}{(\tau_2 - \tau_1)\sigma_1 v_2} \dfrac{W^*(t)}{P(t,\tau_1)G(t,\tau_1)} \\ -\dfrac{v_2(\tau_1 - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_1 - t)]\kappa_2(t)}{(\tau_2 - \tau_1)\sigma_1 v_2} \dfrac{W^*(t)}{P(t,\tau_2)G(t,\tau_2)} \end{pmatrix}$$

and

$$\begin{pmatrix} \hat{\beta}_1^{\mathrm{h}}(t) \\ \hat{\beta}_2^{\mathrm{h}}(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{\tau_2 - t}{\tau_2 - \tau_1} \times \dfrac{\pi I(t)}{P(t,\tau_1)G(t,\tau_1)} \\ \dfrac{\tau_1 - t}{\tau_2 - \tau_1} \times \dfrac{\pi I(t)}{P(t,\tau_2)G(t,\tau_2)} \end{pmatrix}.$$

The optimal spreading demand for forward contracts presented in Eq. (20) is composed of two components:[15] a speculative component denoted by $(\hat{\beta}_1^{\mathrm{s}}(t), \hat{\beta}_2^{\mathrm{s}}(t))$ and a minimum variance hedging component denoted by $(\hat{\beta}_1^{\mathrm{h}}(t), \hat{\beta}_2^{\mathrm{h}}(t))$. Note that there is no Merton/Breeden hedging component which hedges against the fluctuations of the opportunity set. That is, the myopic behavior of the Bernoulli investor remains unchanged in the stochastic interest rates environment.

Let us first analyze the minimum-variance hedging component in Eq. (20). This component serves to offset the risk which stems from the non-traded position. Note that, although in our framework the dynamics of the price of the asset in the non-traded position is driven by one source of uncertainty, the investor must use two forward contracts in order to neutralize the risk resulted from this non-traded position. The dynamics of the forward contracts prices are driven by two sources of uncertainty due to stochastic interest rates. Hence, when the investor includes a forward contract in his portfolio for hedging purposes, he introduces a new source of uncertainty into his portfolio. To offset this new kind of uncertainty, the investor trades another forward contract which is linearly independent of the first one.

Simultaneous positions in two forward contracts with different maturities allow the investor to achieve a perfect hedge (zero instantaneous variance) of his non-traded position. The perfect hedge presented in Eq. (20) involves positions in opposite sign in the two forward contracts. The investor has a short position in the nearby contract and a long position in the deferred contract. Furthermore, the investor has to continuously rebalance his portfolio since the hedging demands of the two forward contracts, $\hat{\beta}_1^{\mathrm{h}}(t)$ and $\hat{\beta}_2^{\mathrm{h}}(t)$, are time dependent. This time dependence is a result of the stochastic interest rates that cause the volatility of the forward contracts to be time dependent although the underlying asset's volatility is not time dependent. The adjusting factors that account for the difference in volatilities are given by the ratios $(\tau_2 - t)/(\tau_2 - \tau_1)$ and $(\tau_1 - t)/(\tau_2 - \tau_1)$ in $\hat{\beta}_1^{\mathrm{h}}(t)$ and $\hat{\beta}_2^{\mathrm{h}}(t)$ respectively. An additional source of time

[15] This is the traditional decomposition which goes back to Anderson and Danthine (1983). For such a decomposition in continuous time see Adler and Detemple (1988a), Adler and Detemple (1988b).

dependence resulted from a price adjusting factors in $\hat{\beta}_1^{\mathrm{h}}(t)$ and $\hat{\beta}_2^{\mathrm{h}}(t)$ given by

$$\frac{\pi I(t)}{P(t,\tau_1)G(t,\tau_1)} \quad \text{and} \quad \frac{\pi I(t)}{P(t,\tau_2)G(t,\tau_2)}$$

respectively.

One may inquire why the forward contracts dynamic spreading strategy is composed of a short position taken in the nearby forward contract and a long position in the distant contract? The economic rationale for this choice stems from the fact that the nearby forward contract has the highest instantaneous correlation with the dynamic of the underlying asset price.[16] In the context of the hedging literature one can view the hedging position in the nearby contract as the traditional offsetting position of the non-traded cash position while the objective of the long position in the deferred contract is to offset any residual risk so as to eliminate completely the instantaneous volatility of the hedged portfolio.

Let us now analyze the speculative component in Eq. (20). This component allows the investor to replicate the growth optimum portfolio. If $v_2(\tau_i - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_i - t)]\kappa_2(t) > 0$, then the speculative component involves a long position in the nearby contract and a short position in the deferred contract. If $v_2(\tau_i - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_i - t)]\kappa_2(t) < 0$, then the speculative component involves a short position in the nearby contract and a long position in the distant contract. Note that $v_2(\tau_i - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_i - t)]\kappa_2(t)$ is the determinant of the volatility matrix of the growth optimum portfolio price dynamics and a forward contract price dynamics. To inquire further into this condition, let us present the following lemma:

Lemma. The dynamics of the forward contracts prices are imperfectly negatively correlated with the growth optimum portfolio dynamics. If $v_2(\tau_i - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_i - t)]\kappa_2(t) > 0$, then the instantaneous correlation coefficient between the forward contracts prices dynamics and the dynamics of the growth optimum portfolio value is an increasing function, in absolute terms, of the forward contracts maturity. If $v_2(\tau_i - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_i - t)]\kappa_2(t) < 0$, then the instantaneous correlation coefficient between the forward contracts prices dynamics and the dynamics of the growth optimum portfolio value is a decreasing function, in absolute terms, of the forward contracts maturities.

[16] To see this, note first that the instantaneous correlation between a futures and the underlying asset is

$$\frac{(\sigma_1 + v_1(\tau_i - t))\sigma_1}{\sigma_1\sqrt{(\sigma_1 + v_1(\tau_i - t))^2 + v_2^2(\tau_i - t)^2}}.$$

Taking the first order derivative with respect to the maturity τ_i leads to the desired result.

Let us analyze the case of $v_2(\tau_i - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_i - t)]\kappa_2(t) > 0$. In this case, the speculative component involves a long position in the nearby contract and a short position in the deferred contract. Using the Lemma, one can show that the deferred contract has the highest instantaneous correlation (in absolute terms) with the growth optimum portfolio. Therefore, a similar interpretation to that was given for the minimum-variance hedging component in Eq. (20) can now be given for the speculative component in Eq. (20). Since the forward contracts price dynamics are negatively correlated with the growth optimum portfolio value dynamics, the investor shorts the deferred contract to arrive at the desired position for replicating his optimal wealth. The residual risk which stems from the short position in the deferred contract is hedged by taking a long position in the nearby forward contract.

When $v_2(\tau_i - t)\kappa_1(t) - [\sigma_1 + v_1(\tau_i - t)]\kappa_2(t) < 0$, then the speculative component involves a short position in the nearby contract and a long position in the deferred contract. Hence, in this case, the forward spreading strategy, which includes both the speculative component and the minimum-variance hedging component, involves a short position in the nearby contract and a long position in the deferred contract.

In the next section, we analyze the spreading strategy of this investor when he faces a futures market.

5. On the effects of marking-to-market

The main difference between forward contracts to futures contracts is the marking to market procedure attributed to the futures contracts. This procedure affects the dynamic spreading strategy of an investor endowed with a non-traded cash position. In this section, we first derive the futures settlement price and then analyze the investor's problem. Finally, we compare the forward spreading strategy to the futures spreading strategy.

The futures contract, written on the asset in the non-traded position (the stock index), is assumed to be continuously marked-to-market. Following Duffie and Stanton (1992), its settlement price at time t equals the price at time t of a cash flow $I(\tau_i)\exp\{\int_t^{\tau_i} r(s)\,\mathrm{d}s\}$ at the instant τ_i, where τ_i is the maturity of the futures contract. Hence,

$$\frac{H(t,\tau_i)}{B(t)} = E^{\tilde{Q}}\left[\frac{I(\tau_i)\exp\{\int_t^{\tau_i} r(s)\,\mathrm{d}s\}}{B(\tau_i)}\middle| F_t\right] \tag{21}$$

and

$$H(t,\tau_i) = E^{\tilde{Q}}[I(\tau_i)|F_t]. \tag{22}$$

As a result, the arbitrage free settlement price of the futures contract is

$$H(t,\tau_i)=\frac{I(t)\exp\{-\delta(\tau_i-t)\}}{P(t,\tau_i)}\exp\left\{(v_1^2+v_2^2)\frac{(\tau_i-t)^3}{3}+\frac{1}{2}\sigma_1 v_1(\tau_i-t)^2\right\}. \tag{23}$$

Since we allow the investor to trade two futures contracts with different maturities and a locally riskless asset, he faces a complete financial market and is able to reach the welfare level of first best optimum as in the case of two forward contracts and a riskless asset. Therefore, this establishes the grounds for comparison of the two optimal strategies which allow the investor to reach the same optimum.

The futures contracts positions are marked-to-market in an interest rate bearing margin account. Denote by $X_i(t)$ the value of the margin account at time t associated with futures contract of maturity τ_i. This value is

$$X_i(t)=\int_0^t \exp\left\{\int_s^t r(s)\,\mathrm{d}s\right\}\beta_i(s)\,\mathrm{d}H(s,\tau_i). \tag{24}$$

The investor's wealth at time t, $W(t)$, is composed of four components: the non-traded position, the two margin accounts associated with the futures positions and the position of the locally riskless asset. Thus,

$$W(t)=\pi I(t)+X_1(t)+X_2(t)+\alpha(t)B(t). \tag{25}$$

The futures contracts dynamic strategy is given in the following proposition:

Proposition 2. The optimal futures trading strategy in the futures contract with maturity τ_1 and the futures contract with maturity τ_2 is, respectively,

$$\begin{pmatrix}\hat{\hat{\beta}}_1(t)\\ \hat{\hat{\beta}}_2(t)\end{pmatrix}=\begin{pmatrix}\hat{\hat{\beta}}_1^{\mathrm{s}}(t)\\ \hat{\hat{\beta}}_2^{\mathrm{s}}(t)\end{pmatrix}+\begin{pmatrix}\hat{\hat{\beta}}_1^{\mathrm{h}}(t)\\ \hat{\hat{\beta}}_2^{\mathrm{h}}(t)\end{pmatrix}, \tag{26}$$

where

$$\begin{pmatrix}\hat{\hat{\beta}}_1^{\mathrm{s}}(t)\\ \hat{\hat{\beta}}_2^{\mathrm{s}}(t)\end{pmatrix}=\begin{pmatrix}\dfrac{v_2(\tau_2-t)\kappa_1(t)-[\sigma_1+v_1(\tau_2-t)]\kappa_2(t)}{(\tau_2-\tau_1)\sigma_1 v_2}\dfrac{W^*(t)}{H(t,\tau_1)}\\[2ex] -\dfrac{v_2(\tau_1-t)\kappa_1(t)-[\sigma_1+v_1(\tau_1-t)]\kappa_2(t)}{(\tau_2-\tau_1)\sigma_1 v_2}\dfrac{W^*(t)}{H(t,\tau_2)}\end{pmatrix}$$

and

$$\begin{pmatrix} \hat{\hat{\beta}}_1^{\mathrm{h}}(t) \\ \hat{\hat{\beta}}_2^{\mathrm{h}}(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{\tau_2 - t}{\tau_2 - \tau_1} \times \dfrac{\pi I(t)}{H(t, \tau_1)} \\ \dfrac{\tau_1 - t}{\tau_2 - \tau_1} \times \dfrac{\pi I(t)}{H(t, \tau_2)} \end{pmatrix}.$$

The dynamic futures spreading strategy has two components: a speculative component denoted by $(\hat{\hat{\beta}}_1^{\mathrm{s}}(t), \hat{\hat{\beta}}_2^{\mathrm{s}}(t))$ and a minimum-variance hedging component denoted by $(\hat{\hat{\beta}}_1^{\mathrm{h}}(t), \hat{\hat{\beta}}_2^{\mathrm{h}}(t))$. Note that there is no Merton/Breeden hedging component to hedge against the fluctuations of the opportunity set. Hence, the marking-to-market procedure does not change the myopic behavior of the Bernoulli investor in the presence of interest rate risk.

Let us first analyze the minimum-variance hedging component. It involves a short position in the nearby futures contract and a long position in the deferred futures contract for similar reasons as explained for the case of forward contracts. This stems from the fact that the forward price dynamics and the futures price dynamics have the same instantaneous volatilities. This is also the reason that the adjusting factors for the difference in volatilities, $(\tau_2 - t)/(\tau_2 - \tau_1)$ and $(\tau_1 - t)/(\tau_2 - \tau_1)$ in $\hat{\hat{\beta}}_1^{\mathrm{h}}(t)$ and $\hat{\hat{\beta}}_2^{\mathrm{h}}(t)$, respectively, are the same as in $\hat{\beta}_1^{\mathrm{h}}(t)$ and $\hat{\beta}_2^{\mathrm{h}}(t)$, respectively. The only difference comes from the price adjustment factors. While in $\hat{\hat{\beta}}_1^{\mathrm{h}}(t)$ and $\hat{\hat{\beta}}_2^{\mathrm{h}}(t)$ they are equal to $\pi I(t)/H(t, \tau_1)$ and $\pi I(t)/H(t, \tau_2)$ respectively, they are equal to $\pi I(t)/P(t, \tau_1)G(t, \tau_1)$ and $\pi I(t)/P(t, \tau_2)G(t, \tau_2)$ in $\hat{\beta}_1^{\mathrm{h}}(t)$ and $\hat{\beta}_2^{\mathrm{h}}(t)$, respectively. Hence, while in the case of futures contracts, one adjusts with respect to the settlement price of the futures contracts, in the case of forward contracts one adjusts with respect to the present value of the forward price.

The speculative component of the futures spreading strategy in Eq. (26) has the same intuitive content as the forward contracts spreading strategy's speculative component in Eq. (20).

The following lemma compares the optimal forward spreading strategy and the optimal futures spreading strategy by writing the second in terms of the first. In the hedging literature, the term that multiplies the forward strategy in order to obtain the futures strategy is called the tailing factor. However, explicit expression for the tailing factor are available[17] only when interest rates are deterministic and for an investor interested to hedge his non-traded position. In the following lemma, we give the tailing factor when interest rates are stochastic for an investor who seeks to maximize the expected utility of his terminal wealth.

[17] See Figlewski et al. (1991). A recent paper by Lioui (1997) addresses this issue in the case of interest rates derivatives.

Corollary. The relationship between the two legs of the optimal spreading strategy using futures contracts and the corresponding two legs using forward contracts is as follows:[18]

$$\hat{\hat{\beta}}_1(t) = \mathrm{TF}_1(t) \times \hat{\beta}_1(t), \tag{27}$$

$$\hat{\hat{\beta}}_2(t) = \mathrm{TF}_2(t) \times \hat{\beta}_2(t), \tag{28}$$

where

$$\mathrm{TF}_i(t) = P(t, \tau_i) \times \exp\left\{ -(v_1^2 + v_2^2)\frac{(\tau_i - t)^3}{3} - \frac{1}{2}\sigma_1 v_1 (\tau_i - t)^2 \right\}.$$

The corollary implies that: (i) there exist two tailing factors, each one for each leg of the spreading strategy; (ii) the tailing factors are independent of the individuals characteristics; in particular they are independent of the investor's horizon (this is a 'separation' result); (iii) the number of futures contracts held by the investor is always smaller, in absolute terms, than the corresponding position for forward contracts.

The tailing factors are composed of two terms. A discount bond whose maturity is equal to the maturity of the futures contract. Note that this term is the one which appears in a deterministic interest rates environment. The second term depends explicitly upon interest rates volatility. Moreover, this second term is exactly the ratio of the forward contract price to the futures contract price. This does not come as a surprise since the difference between the two optimal spreading strategies stems from the price adjusting factor.

6. Extensions

Up to this point we have based our analysis on the assumption of complete markets. This was necessary in order to preserve the simplicity of our results. It also allowed us to show in the previous section the effect of marking-to-market on the investor's strategy.

The purpose of this section is twofold. First, we will show how our results are affected when the assumption of complete markets is relaxed. Second, we wish to deviate from intra-market spreading which has been the focus of our attention and examine our results in light of inter-market spreading.

The problem to be solved can be simplified by focusing on a pure hedger, namely an investor who is infinitely risk averse and whose sole purpose is to trade derivative assets in order to minimize the risk stemming from his terminal

[18] This can be shown directly by comparing Eqs. (20) and (26).

wealth. We limit ourselves to forward contracts and forgo the opportunity to show the effect of marking-to-market in incomplete markets.

We extend our framework to an incomplete market by adding an additional source of uncertainty which is specific to the stock index. This appears to us to be the most natural approach since the term structure of interest rates is already affected by a specific source of risk. Thus, assuming that the primitive assets market includes the same locally riskless asset as Eq. (2) and the same discount bond as Eq. (4), the dynamics of the stock index price process is as follows:

$$\mathrm{d}I(t) = (\psi(t, I(t)) - \delta)I(t)\,\mathrm{d}t + \sigma_1 I(t)\,\mathrm{d}Z_1(t) + \sigma_3 I(t)\,\mathrm{d}Z_3(t), \quad I(0) = x > 0. \tag{5'}$$

We assume that forward contracts written on the stock index are traded with a price at time t for a contract with maturity τ_i as follows:

$$G(t, \tau_i) = \frac{I(t)\exp\{-\delta(\tau_i - t)\}}{P(t, \tau_i)}. \tag{29}$$

Although markets are incomplete any forward contract with a maturity $\tau_i < \tau_D$ can be replicated using a suitable strategy. This strategy involves trading the locally riskless asset, the stock index and a discount bond which matures at the same time as the forward contract. Such a discount bond is not traded. However, it can be replicated using the existing discount bond of maturity τ_D and the locally riskless asset.

In such a market, only two forward contracts with different maturity dates are traded. Any additional forward contract would be redundant because it can be replicated using the two contracts that are already being traded. A simple way to show this is based on the results found by Harrison and Pliska (1981) whereby the asset market for diffusion price processes is complete when the volatility matrix of the traded assets is invertible. Now, assuming that three stock index forward contracts are traded with maturities τ_1, τ_2 and τ_3 such that $\tau_1 < \tau_2 < \tau_3 < \tau_D$, we can apply Ito's lemma to Eq. (29) and get the dynamics of the price process of each contract as follows:

$$\begin{aligned} \mathrm{d}G(t, \tau_i) = (\cdot)\,\mathrm{d}t + G(t, \tau_i)[(\sigma_1 + v_1(\tau_i - t))\,\mathrm{d}Z_1(t) \\ + v_2(\tau_i - t)\,\mathrm{d}Z_2(t) + \sigma_3\,\mathrm{d}Z_3(t)]. \end{aligned} \tag{30}$$

We can construct the volatility matrix as follows:

$$\begin{pmatrix} \sigma_1 + v_1(\tau_1 - t) & v_2(\tau_1 - t) & \sigma_3 \\ \sigma_1 + v_1(\tau_2 - t) & v_2(\tau_2 - t) & \sigma_3 \\ \sigma_1 + v_1(\tau_3 - t) & v_2(\tau_3 - t) & \sigma_3 \end{pmatrix}. \tag{31}$$

It is easy to show that Eq. (31) has a rank of 2 and therefore is not invertible.

We assume that forward contracts written on the discount bond are also traded. The price at time t of such forward contracts maturing at time τ_i is

$$g(t,\tau_i) = \frac{P(t,\tau_D)}{P(t,\tau_i)}. \tag{32}$$

Following the previous analysis, one can show that only one interest rate forward contract will be traded.

The hedger has the option of two trading strategies. In both strategies he will use only two forward contracts. He can either use two forward contracts written on the stock index (intra-market spreading) or one stock index forward contract and an interest rate forward contract (inter-market spreading). A third contract will always be redundant and therefore our hedger will always be trading in incomplete markets.

We first turn to examine the case of intra-market spreading. Assume an investor who is endowed with a non-traded cash position of $\pi > 0$ units of the stock index. The investor trades continuously two stock index forward contracts and the locally riskless asset. The investor's optimal strategy is chosen such that the volatility of his portfolio is minimized. He follows an admissible and self financing trading strategy. Therefore, at each time t, the value of his portfolio is given as follows:

$$\begin{aligned} W(t) = \pi I(t) + \int_0^t \pi\delta I(s)\,\mathrm{d}s + P(t,\tau_1)\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1) \\ + P(t,\tau_2)\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2) + \alpha(t)B(t) \end{aligned} \tag{33}$$

and then,

$$\begin{aligned} \mathrm{d}W(t) = \pi\,\mathrm{d}I(t) + \pi\delta I(t)\,\mathrm{d}t + \mathrm{d}\left(P(t,\tau_1)\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)\right) \\ + \mathrm{d}\left(P(t,\tau_2)\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2)\right) + \alpha(t)\,\mathrm{d}B(t). \end{aligned} \tag{34}$$

Now, substituting for Eqs. (5′) and (30) in Eq. (34), we get

$$\begin{aligned} \mathrm{d}W(t) &= (\cdot)\mathrm{d}t \\ &+ \begin{bmatrix} \pi I(t)\sigma_1 + \beta_1(t)P(t,\tau_1)G(t,\tau_1)(\sigma_1 + v_1(\tau_1 - t)) - P(t,\tau_1)\left(\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)\right)v_1(\tau_1 - t) \\ + \beta_2(t)P(t,\tau_2)G(t,\tau_2)(\sigma_1 + v_1(\tau_2 - t)) - P(t,\tau_2)\left(\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2)\right)v_1(\tau_2 - t) \end{bmatrix}\mathrm{d}Z_1(t) \end{aligned}$$

$$+\left[\begin{array}{l}\beta_1(t)P(t,\tau_1)G(t,\tau_1)v_2(\tau_1-t)-P(t,\tau_1)\left(\int_0^t\beta_1(s)\,\mathrm{d}G(s,\tau_1)\right)v_2(\tau_1-t)\\ +\beta_2(t)P(t,\tau_2)G(t,\tau_2)v_2(\tau_2-t)-P(t,\tau_2)\left(\int_0^t\beta_2(s)\,\mathrm{d}G(s,\tau_2)\right)v_2(\tau_2-t)\end{array}\right]\mathrm{d}Z_2(t)$$
$$+\left[\pi I(t)\sigma_3+\beta_1(t)P(t,\tau_1)G(t,\tau_1)\sigma_3+\beta_2(t)P(t,\tau_2)G(t,\tau_2)\sigma_3\right]\mathrm{d}Z_3(t). \tag{35}$$

One can easily show that the following strategy allows the investor to achieve a perfect hedge:

$$\begin{aligned}\beta_1(t)&=-\frac{\tau_2-t}{\tau_2-\tau_1}\times\frac{\pi I(t)}{P(t,\tau_1)G(t,\tau_1)},\\ \beta_2(t)&=\frac{\tau_1-t}{\tau_2-\tau_1}\times\frac{\pi I(t)}{P(t,\tau_2)G(t,\tau_2)}.\end{aligned} \tag{36}$$

This strategy has the following characteristics. First, a constrained investor who uses this strategy can achieve a prefect hedge even when markets are not complete. Second, the strategy involves a short position in the nearby contract and a long position in the deferred contract. Intuitively, this can be explained along the lines of the previous section. Finally, if the dividend yield is equal to zero then this strategy is identical to the one that would have been chosen had markets been complete.

Turning now to inter-market spreading we assume that the investor trades on stock index forward contracts and interest rates forward contracts. For simplicity and without loss of generality, we assume that both forward contracts have the same maturity date. In such a case the investor's wealth at each time t is:

$$\begin{aligned}W(t)={}&\pi I(t)+\int_0^t\pi\delta I(s)\,\mathrm{d}s+P(t,\tau_1)\int_0^t\beta(s)\,\mathrm{d}G(s,\tau_1)\\ &+P(t,\tau_1)\int_0^t\theta(s)\,\mathrm{d}g(s,\tau_1)+\bar{\alpha}(t)B(t),\end{aligned} \tag{37}$$

where $\beta(t)$ is the number of stock index forward contracts held at time t and $\theta(t)$ is the number of interest rate forward contracts held at time t. We then have

$$\begin{aligned}\mathrm{d}W(t)={}&\pi\mathrm{d}I(t)+\pi\delta I(t)\,\mathrm{d}t+\mathrm{d}\left(P(t,\tau_1)\int_0^t\beta(s)\,\mathrm{d}G(s,\tau_1)\right)\\ &+\mathrm{d}\left(P(t,\tau_1)\int_0^t\theta(s)\,\mathrm{d}g(s,\tau_1)\right)+\bar{\alpha}(t)\,\mathrm{d}B(t).\end{aligned} \tag{38}$$

Applying Ito's lemma to Eq. (32), we get the dynamics of the interest rates forward contract price process,

$$\mathrm{d}g(t,\tau_1)=g(t,\tau_1)[(\cdot)\,\mathrm{d}t-v_1(\tau_D-\tau_1)\,\mathrm{d}Z_1(t)-v_2(\tau_D-\tau_1)\,\mathrm{d}Z_2(t)]. \tag{39}$$

Substituting Eqs. (5′) and (39) in Eq. (38) yields:

$$
\begin{aligned}
\mathrm{d}W(t) &= (\cdot)\,\mathrm{d}t \\
&+ \left[\begin{array}{c}
\pi I(t)\sigma_1 + \beta(t)P(t,\tau_1)G(t,\tau_1)(\sigma_1 + v_1(\tau_1 - t)) - P(t,\tau_1)\left(\displaystyle\int_0^t \beta(s)\,\mathrm{d}G(s,\tau_1)\right)v_1(\tau_1 - t) \\
- \theta(t)P(t,\tau_1)g(t,\tau_1)v_1(\tau_\mathrm{D} - \tau_1) - P(t,\tau_1)\left(\displaystyle\int_0^t \theta(s)\,\mathrm{d}g(s,\tau_1)\right)v_1(\tau_1 - t)
\end{array}\right]\mathrm{d}Z_1(t) \\
&+ \left[\begin{array}{c}
\beta(t)P(t,\tau_1)G(t,\tau_1)v_2(\tau_1 - t) - P(t,\tau_1)\left(\displaystyle\int_0^t \beta(s)\,\mathrm{d}G(s,\tau_1)\right)v_2(\tau_1 - t) \\
- \theta(t)P(t,\tau_1)g(t,\tau_1)v_2(\tau_\mathrm{D} - \tau_1) - P(t,\tau_1)\left(\displaystyle\int_0^t \theta(s)\,\mathrm{d}g(s,\tau_1)\right)v_2(\tau_1 - t)
\end{array}\right]\mathrm{d}Z_2(t) \\
&+ [\pi I(t)\sigma_3 + \beta(t)P(t,\tau_1)G(t,\tau_1)\sigma_3]\,\mathrm{d}Z_3(t).
\end{aligned}
\tag{40}
$$

One can show that the following strategy leads to a perfect hedge:

$$
\begin{aligned}
\beta(t) &= -\frac{I(t)}{P(t,\tau_1)G(t,\tau_1)}\pi, \\
\theta(t) &= -\frac{I(t)}{P(t,\tau_\mathrm{D})}\frac{\tau_1 - t}{\tau_\mathrm{D} - \tau_1}\pi.
\end{aligned}
\tag{41}
$$

This strategy requires the investor to short the stock index forward contract as well as the interest rates forward contract. Interestingly, a minimum-variance component does not appear in the stock index forward leg. Furthermore, when there are no dividends, the position in the stock index forward contract becomes the usual static position $\beta(t) = -\pi$. Note, that our result (41) does not allow zero interest rates risk ($v_1 = v_2 = 0$). This is because the solution to the investor's problem assumes non-zero interest rates risk. Therefore, the usual forward strategy when interest rates are deterministic cannot be obtained from Eq. (41).

7. Concluding remarks

This paper assumes an investor who has a non-traded position operating in a stochastic interest rates environment. The investor uses either distinct futures contracts or distinct forward contracts in order to maximize his expected utility of terminal wealth. Due to stochastic interest rates, there is an imperfect correlation between the dynamics of either the futures settlement prices or the forward prices with the underlying asset price dynamics. Therefore, in order to reach the welfare level of the first best optimum the investor must trade either two distinct futures contracts or two distinct forward contracts.

The optimal forward contracts spreading strategy has two components, a speculative component and a minimum-variance hedging component. The investor does not trade derivative contracts in order to hedge against the fluctuations of the opportunity set. The minimum-variance hedging component is composed of a short position in the nearby contract and a long position in the deferred contract. The hedging component serves to offset the risk which stems from the investor's non-traded cash position. The speculative component serves to replicate the growth optimum portfolio. The speculative component is composed of a short position in the contract which is the most negatively correlated with the growth optimum portfolio and a long position in the other contract. The marking-to-market procedure of the futures positions forces the investor to hold less futures contracts that in the corresponding forward contract positions.

The analysis has been extended to incomplete markets where it is shown that using two forward contracts still allows the hedger to achieve a perfect hedge even though the economy is driven by three sources of uncertainty.

The design of the futures contracts in our framework is given exogeneously. A natural extension of our work would be to find an equilibrium based explanation to the existence of such a structure. Some papers, using a partial equilibrium approach, address the issue of futures innovation. The work by Duffie and Jackson (1989) and Cuny (1993) concerning futures innovation in a market with frictions concludes that only futures contracts with orthogonal payoffs will be created since contract creation is costly and correlated contracts appear to consume resources without expanding the hedging opportunities available to the investor. The work of Tashjian and Weissman (1995) work shows, in a static framework, that when the fee is endogenised, it can be optimal to create futures contracts with correlated payoffs. Our intertemporal setting suggests that it would also be optimal to create futures contracts with correlated payoffs when the futures contracts are written on the same underlying asset with different maturities. Kamara (1993) dealt with the designing of the structure of forward maturities. However, in his framework forward contracts that have different maturities are perfectly correlated and thus can be used only for roll-over purposes.

Acknowledgements

We would like to thank Avraham Kamara, Patrice Poncet, Elisabeth Tashjian, Zvi Wiener and Participants at The 7th Symposium on Money, Banking and Finance, Karlsrhue December 1996. Two anonymous referees provided valuable remarks and suggestions which greatly improved the presentation of the paper. Miriam Krausz provided excellent editorial assistance. All errors are ours.

Appendix A.

A.1. Proof of Proposition 1

The martingale approach[19] is used to solve this problem. One need not construct a martingale measure on the investor's financial market (derivatives assets and riskless asset) since such a market is complete and thus the pricing function will be the martingale measure constructed in Eq. (9). Note, due to market completeness, we can use the martingale approach to solve for our problem even though the investor's is endowed with a random terminal wealth (the constrained position). One can see the investor's problem 'as if ' he was endowed with an initial wealth whose initial value is the initial value of the non-traded position.

The investor's program becomes

$$\max_{W(\tau_I)} \quad E^{Q}[\mathrm{Ln}(W(\tau_I))]$$

$$\text{s.t.} \quad E^{\tilde{Q}}\left[\frac{W(\tau_I)}{B(\tau_I)}\right] = \pi I(0)\exp\{-\delta\tau_I\}.$$

Using Cox and Huang (1991) (Proposition 4.2 p. 477), one can easily see that this program has a unique solution. Using the Lagrangian theory it follows that the solution to the preceding program is such that

$$\frac{1}{W^*(\tau_I)} - \lambda\frac{1}{B(\tau_I)}\eta(\tau_I) = 0, \tag{A.1}$$

where we have used the fact that $E^{\tilde{Q}}\,[Y(\tau_I)] = E^{Q}[Y(\tau_I)\eta(\tau_I)]$. Thus,

$$W^*(\tau_I) = \frac{1}{\lambda}B(\tau_I)\eta^{-1}(\tau_I), \tag{A.2}$$

where $\lambda > 0$ is the Lagrangian multiplier associated with the budget constraint. Using Eq. (A.2) and the budget constraint one obtains

$$W^*(\tau_I) = \pi I(0)\exp\{-\delta\tau_I\}B(\tau_I)\eta^{-1}(\tau_I) \tag{A.3}$$

By construction of the martingale measure $\tilde{Q}$, the value at each time t of this investor's wealth is

$$\frac{W^*(t)}{B(t)} = E^{\tilde{Q}}\left[\left.\frac{W^*(\tau_I)}{B(\tau_I)}\right|\boldsymbol{F}_t\right]. \tag{A.4}$$

[19] For an excellent account see Duffie (1996) (Chapter 10 p. 219). See also the seminal papers by Karatzas et al. (1987) and Cox and Huang (1989), Cox and Huang (1991).

Using Cox and Huang (1989) (Lemma 2.5 p. 43) (Bayes rule), Eq. (A.4) becomes

$$\frac{W^*(t)}{B(t)} = \eta^{-1}(t)E^Q\left[\frac{W^*(\tau_I)}{B(\tau_I)}\eta(\tau_I)\middle|\boldsymbol{F}_t\right]. \tag{A.5}$$

Substituting for the investor's wealth from Eq. (A.3) yields

$$W^*(t) = \pi I(0)\exp\{-\delta\tau_I\}B(t)\eta^{-1}(t). \tag{A.6}$$

Applying Ito's lemma to Eq. (A.6), one obtains

$$\mathrm{d}W^*(t) = (\cdot)\,\mathrm{d}t - W^*(t)\kappa_1(t)\,\mathrm{d}Z_1(t) - W^*(t)\kappa_2(t)\,\mathrm{d}Z_2(t). \tag{A.7}$$

This wealth is also generated by a self-financing strategy satisfying Eq. (17), and thus satisfies

$$\mathrm{d}W^*(t) = \pi\,\mathrm{d}I(t) + \pi\delta I(t)\,\mathrm{d}t + \mathrm{d}\left(P(t,\tau_1)\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)\right)$$
$$+ \mathrm{d}\left(P(t,\tau_2)\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2)\right) + \alpha(t)\,\mathrm{d}B(t). \tag{A.8}$$

Applying Ito's lemma to Eq. (16), one obtains the dynamics of the forward contract price as follows:

$$\mathrm{d}G(t,\tau_i) = (\cdot)\,\mathrm{d}t + G(t,\tau_i)[(\sigma_1 + v_1(\tau_i - t))\,\mathrm{d}Z_1(t)$$
$$+ v_2(\tau_i - t)\,\mathrm{d}Z_2(t)]. \tag{A.9}$$

Hence, substituting for the stock index price dynamics from Eq. (5) and for the forward price dynamics from Eq. (A.9) in Eq. (A.8), one obtains

$$\begin{aligned}\mathrm{d}W^*(t) &= (\cdot)\,\mathrm{d}t\\ &+\begin{bmatrix}\pi I(t)\sigma_1 + \beta_1(t)P(t,\tau_1)G(t,\tau_1)(\sigma_1 + v_1(\tau_1 - t)) - P(t,\tau_1)\left(\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)\right)v_1(\tau_1 - t)\\ + \beta_2(t)P(t,\tau_2)G(t,\tau_2)(\sigma_1 + v_1(\tau_2 - t)) - P(t,\tau_2)\left(\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2)\right)v_1(\tau_2 - t)\end{bmatrix}\mathrm{d}Z_1(t)\\ &+\begin{bmatrix}\beta_1(t)P(t,\tau_1)G(t,\tau_1)v_2(\tau_1 - t) - P(t,\tau_1)\left(\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)\right)v_2(\tau_1 - t)\\ + \beta_2(t)P(t,\tau_2)G(t,\tau_2)v_2(\tau_2 - t) - P(t,\tau_2)\left(\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2)\right)v_2(\tau_2 - t)\end{bmatrix}\mathrm{d}Z_2(t).\end{aligned} \tag{A.10}$$

By equating the diffusion terms from Eqs. (A.7) and (A.10), one obtains

$$\begin{aligned}
&\left[\begin{array}{c}\pi I(t)\sigma_1 + \beta_1(t)P(t,\tau_1)G(t,\tau_1)(\sigma_1 + v_1(\tau_1 - t)) - P(t,\tau_1)\left(\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)\right)v_1(\tau_1 - t)\\
+ \beta_2(t)P(t,\tau_2)G(t,\tau_2)(\sigma_1 + v_1(\tau_2 - t)) - P(t,\tau_2)\left(\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2)\right)v_1(\tau_2 - t)\end{array}\right]\\
&= -W^*(t)\kappa_1(t)\\
&\left[\begin{array}{c}\beta_1(t)P(t,\tau_1)G(t,\tau_1)v_2(\tau_1 - t) - P(t,\tau_1)\left(\int_0^t \beta_1(s)\,\mathrm{d}G(s,\tau_1)\right)v_2(\tau_1 - t)\\
+ \beta_2(t)P(t,\tau_2)G(t,\tau_2)v_2(\tau_2 - t) - P(t,\tau_2)\left(\int_0^t \beta_2(s)\,\mathrm{d}G(s,\tau_2)\right)v_2(\tau_2 - t)\end{array}\right]\\
&= -W^*(t)\kappa_2(t)
\end{aligned} \tag{A.11}$$

and solving Eq. (A.11) leads to the desired result.

A.2. *Proof of the lemma*

The instantaneous correlation coefficient between the dynamics of the growth optimum portfolio value and the dynamics of a forward contract, whose maturity is τ_i, price dynamics is

$$\frac{-(\sigma_1 + v_1(\tau_i - t))\kappa_1(t) - v_2(\tau_i - t)\kappa_2(t)}{\sqrt{\kappa_1^2(t) + \kappa_2^2(t)}\sqrt{(_1 + v_1(\tau_i - t))^2 + (v_2(\tau_i - t))^2}},$$

where we have used Eqs. (A.7) and (A.9). Taking the first-order derivative with respect to the maturity τ_i one obtains

$$\frac{\sigma_1 v_2[v_2(\tau_i - t)\kappa_1(t) - (\sigma_1 + v_1(\tau_i - t))\kappa_2(t)]}{(\kappa_1^2(t) + \kappa_2^2(t))((\sigma_1 + v_1(\tau_i - t))^2 + (v_2(\tau_i - t))^2)^{3/2}}$$

and the result follows.

A.3. *Proof of Proposition 2*

Since the financial market which is composed of two futures contracts and the locally riskless asset is complete, the investor's will reach the welfare level of the first best optimum. Thus, the investor's optimal wealth is given by Eq. (A.3) and its dynamics is given by Eq. (A.7).

When the investor trades future contracts, his wealth satisfies the self financing property too. Hence, from Eq. (25), the investor's wealth dynamics is

$$\mathrm{d}W^*(t) = \pi\,\mathrm{d}I(t) + \pi\delta I(t)\,\mathrm{d}t + \mathrm{d}X_1(t) + \mathrm{d}X_2(t) + \alpha(t)\,\mathrm{d}B(t). \tag{A.12}$$

Applying Ito's lemma to Eqs. (24) and (23), one obtains, respectively,

$$\mathrm{d}X_i(t) = r(t)X_i(t)\,\mathrm{d}t + \beta_i(t)\,\mathrm{d}H(t,\tau_i), \tag{A.13}$$

$$\mathrm{d}H(t,\tau_i) = (\cdot)\,\mathrm{d}t + H(t,\tau_i)[(\sigma_1 + v_1(\tau_i - t))\,\mathrm{d}Z_1(t) + v_2(\tau_i - t)\,\mathrm{d}Z_2(t)]. \tag{A.14}$$

Substituting for Eqs. (A.13) and (A.14) in Eq. (A.12), one obtains

$$\begin{aligned}\mathrm{d}W^*(t) = (\cdot)\,\mathrm{d}t &+ [\pi I(t)\sigma_1 + \hat{\hat{\beta}}_1(t)H(t,\tau_1)(\sigma_1 + v_1(\tau_1 - t)) \\ &+ \hat{\hat{\beta}}_2(t)H(t,\tau_2)(\sigma_1 + v_1(\tau_2 - t))]\,\mathrm{d}Z_1(t) \\ &+ [\hat{\hat{\beta}}_1(t)H(t,\tau_1)v_2(\tau_1 - t) + \hat{\hat{\beta}}_2(t)H(t,\tau_2)v_2(\tau_2 - t)]\,\mathrm{d}Z_2(t).\end{aligned} \tag{A.15}$$

Equating the diffusion terms in Eqs. (A.7) and (A.15), one obtains

$$\begin{aligned}&\pi I(t)\sigma_1 + \hat{\hat{\beta}}_1(t)H(t,\tau_1)(\sigma_1 + v_1(\tau_1 - t)) + \hat{\hat{\beta}}_2(t)H(t,\tau_2)(\sigma_1 + v_1(\tau_2 - t)) \\ &\quad = -W^*(t)\kappa_1(t) \\ &\hat{\hat{\beta}}_1(t)H(t,\tau_1)v_2(\tau_1 - t) + \hat{\hat{\beta}}_2(t)H(t,\tau_2)v_2(\tau_2 - t) = -W^*(t)\kappa_2(t).\end{aligned} \tag{A.16}$$

and solving Eq. (A.16) leads to the desired result.

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